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Vector States for Single and Multiple-Pole Resonances

M. Gadella[1]

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The formulation of quantum mechanics in rigged Hilbert spaces is used to study the vector states for resonance states or Gamow vectors. An important part of the work is devoted to the construction of Gamow vectors for resonances that appear as multiple poles on the analytic continuation of the *S*-matrix, *S*(*E*). The kinematical behavior of these vectors is also studied. This construction allow for generalized spectral decompositions of the Hamiltonian and the evolutionary semigroups, valid on certain locally convex spaces. Also a first attempt is made to define the resonance states as densities in an extension of the Liouville space, here called rigged Liouville space.

1. INTRODUCTION

This is a paper on resonances and the representation of resonance states on certain extensions of Hilbert spaces called rigged Hilbert spaces (RHS). A RHS is a triplet of spaces:

$$\mathbf{\Phi} \subset \mathcal{H} \subset {}^{\times}\mathbf{\Phi} \tag{1.1}$$

where $\mathcal{H}$ is a Hilbert space. The space $\mathbf{\Phi}$ is a dense subspace of $\mathcal{H}$ endowed with its own topology (stronger than the topology it has as a subspace of $\mathcal{H}$) and ${}^{\times}\mathbf{\Phi}$ the antidual of $\mathbf{\Phi}$, i.e., the space of continuous antilinear mappings from $\mathbf{\Phi}$ into C, the set of complex numbers. This antidual also has a natural topology so that the spaces $\mathbf{\Phi}$ and ${}^{\times}\mathbf{\Phi}$ are a dual pair in the sense that $\mathbf{\Phi}$ is the antidual of ${}^{\times}\mathbf{\Phi}$. This happens in particular if $\mathbf{\Phi}$ possesses the property of nuclearity. For details on RHS and their mathematical properties see Gel'fand and Vilenkin (1964), Maurin (1968), Roberts (1966), Antoine (1969), and Melsheimer (1974).

[1] Departamento de Física Teórica, Facultad de Ciencias, Universidad de Valladolid, 47011 Valladolid, Spain.

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RHS have appeared in the mathematical literature in connection with the theory of representations of noncompact groups (Gel'fand and Vilenkin, 1964; Maurin, 1968). It was realized soon that RHS are suitable for making rigorous the Dirac formalism of quantum mechanics (Dirac, 1965) (Hilbert spaces do not allow for plane waves, localized states, and other nonnormalizable states or complete sets of eigenvalues for most observables) (Bohm, 1965; Roberts, 1966; Antoine, 1969; Melsheimer, 1974). In particular, the requirement that an observable must have a complete set of proper eigenvectors can only be implemented on a rigorous basis in a suitable RHS.

Another limitation of quantum mechanics on Hilbert space is that it does not allow for Gamow states (Gamow, 1928) or states describing resonance states, since they are also not normalizable. The formulation of quantum mechanics on RHS permits the use of these nonnormalizable entities as elements of the antidual space $^{\times}\mathbf{\Phi}$.

Another very interesting property is the possibility of extending by continuity certain (bounded or not) operators on $\mathcal{H}$ to the antiduals of some spaces $\mathbf{\Phi}$ such that $\mathbf{\Phi} \subset \mathcal{H} \subset {}^{\times}\mathbf{\Phi}$ is a RHS. In fact, if A is an operator on $\mathcal{H}$ such that (1) for any $\varphi \in \mathbf{\Phi}$, $A\varphi \in \mathbf{\Phi}$, and (2) A is continuous on $\mathbf{\Phi}$ (with respect to the topology on $\mathbf{\Phi}$), then A can be extended to a continuous operator on $^{\times}\mathbf{\Phi}$ by using the following definition:

$$\langle A\varphi | F\rangle = \langle \varphi | AF\rangle \tag{1.2}$$

where $\langle \varphi | F\rangle$ denotes the action of the functional $F \in {}^{\times}\mathbf{\Phi}$ on the vector $\varphi \in \mathbf{\Phi}$ (in this paper we shall use the same symbol for an operator and for its extension to a bigger space). In particular, if A is self-adjoint, we always can find a $\mathbf{\Phi}$ with the above properties. For details, see Gel'fand and Vilenkin (1964), Maurin (1968), Roberts (1966), Antoine (1969), and Melsheimer (1974).

In the formalism of the S-matrix, resonances appear as poles of the analytically continued S-matrix in the energy representation to the second sheet of a two-sheeted Riemann surface corresponding to the transformation $p = \sqrt{E}$ (Bohm, 1994). Bohm (1980, 1981) first constructed rigorously the Gamow vectors corresponding to simple pole resonances of this continuation of the S-matrix. In addition, the unitary group giving the time evolution splits into two semigroups in the presence of resonances. Using this fact, Prigogine and co-workers (Prigogine, 1992; Petrovski and Prigogine, 1991; Antoniou and Prigogine, 1993) realized that resonances exhibit a typical irreversible behavior at the microphysical level.

The possibility of the existence of resonances as poles of the continuation of the S-matrix of order higher than one has also been considered (Bohm, 1994; Newton, 1982). In connection with this, Jordan block structures have been used for the Hamiltonian. These Jordan block structures can be easily derived and understood in the context of the formulation of quantum mechanics on RHS. For this formulation, we refer the reader to (Bohm, 1965, 1994; Bohm and Gadella, 1989).

The idea of exploring Gamow densities or density operators has not yet been much used. Here we make a preliminary attempt in this direction.

This paper is organized as follows: In the next section, we briefly review the fundamentals on the construction of Gamow vectors for single and multiple-pole resonances. Then we proceed to the generalization to multiple-pole resonances. In Section 3, we study generalized spectral decompositions for the total Hamiltonian and also for the semigroups given the time evolution. In the last part of the paper, we present the idea of Gamow states for resonances and make an attempt to define their time evolution. General ideas are presented in Section 4 and the time evolution for Gamow states for multiple pole resonances is treated in Section 5.

2. MULTIPLE-POLE RESONANCES

In the present and next sections, we study a simple model of resonant scattering, which has been studied elsewhere (Bohm, 1980, 1981, 1994; Bohm and Gadella, 1989; Bohm *et al.*, 1997), in which we assume further properties on the poles of the S-matrix. We do not discuss here the details of resonant scattering processes, which are presented, for instance, in Bohm (1994). Here, we assume that resonances are characterized by poles of the S-matrix, in the energy representation, on the second sheet of the two-sheeted Riemann surface associated with the transformation $p = \sqrt{E}$. As in the above references, we assume these poles come in conjugate pairs having positive real part and nonzero imaginary part (usually associated with resonant energy and mean life, respectively). For the sake of simplicity, we further assume that we have one such pair only. In our previous work on the subject, we supposed that these resonance poles are simple. In this paper, we drop this assumption, although we admit that any pair of conjugate poles are of the same order. This is consistent with the properties of the S-matrix (Newton, 1982; Nussenszveig, 1972).

To understand our notation [which has been explained in Bohm and Gadella (1989)], we assume that the continuous parts of the "free" Hamiltonian K and the "perturbed" Hamiltonian $H = K + V$ are unitarily equivalent. This equivalence is usually given by the Møller wave operators $\Omega^{\pm}$. We may further assume that their common continuous spectrum is absolutely continuous and given by $\mathbb{R}^+ = [0, \infty)$. Although this is not essential, it simplifies the discussion. Thus, H and K are unitarily equivalent to the multiplication operator $\hat{E}$ on $L^2(\mathbb{R}^+)$ $[\hat{E}\varphi(E) = E\varphi(E)]$.

Henceforth, we call $\mathcal{H}_{ac}(K)$ and $\mathcal{H}_{ac}(H)$ the absolutely continuous spaces of K and H, respectively (Read and Simon, 1972; Amrein *et al.*, 1977). Let U be a unitary operator that "diagonalizes" the restriction of K to $\mathcal{H}_{ac}(K)$ ($UKU^{-1} = \hat{E}$). One has that $U(\boldsymbol{\Omega}^{\pm})^{-1}H(\boldsymbol{\Omega}^{\pm})U^{-1} = \hat{E}$, i.e., $(\boldsymbol{\Omega}^{\pm})U^{-1}$ "diagonalize" the restriction of H to $\mathcal{H}_{ac}(H)$. Thus, we have the following diagram:

$$\mathcal{H}_{ac}(H) \xrightarrow{(\boldsymbol{\Omega}^{\pm})^{-1}} \mathcal{H}_{ac}(K) \xrightarrow{U} L^2(\ ^+) \tag{2.1}$$

Consider now the set of spaces $\mathcal{H}^2_{\pm} \cap S$, where $\mathcal{H}_{\pm}$ are the spaces of Hardy class functions on the $\left\{\begin{matrix}\text{upper}\\ \text{lower}\end{matrix}\right\}$ half-plane, respectively (Duren, 1970; Koosis, 1980), and S is the Schwartz space of all functions admitting derivatives at all orders such that they and their derivatives go to zero faster than the inverse of any polynomial at infinity (Reed and Simon, 1972). The restriction of any function on $\mathcal{H}^2_{\pm} \cap S$ to R^+ uniquely determines its values on the whole real line (van Winter, 1974). These restrictions form dense subspaces of $L^2(\mathrm{R}^+)$ (Gadella, 1983) and determine two one-to-one onto mappings:

$$\theta_{\pm}:\ \ \mathcal{H}^2_{\pm} \cap S \mapsto \mathcal{H}^2_{\pm} \cap S|_{R_+} = \Delta_{\pm} \tag{2.2}$$

$\mathcal{H}^2_{\pm} \cap S$ are Fréchet nuclear spaces (Gel'fand and Vilenkin, 1964; Bohm and Gadella, 1989; Schaeffer, 1970). This structure is transmitted to $\Delta_{\pm}$ via $\theta_{\pm}$. Also, the identity mapping I: $\Delta_{\pm} \mapsto L^2(\mathrm{R}^+)$ is continuous (Bohm and Gadella, 1989). Therefore, if ${}^{\times}(\Delta_{\pm})$ represent the antidual spaces of $\Delta_{\pm}$ (spaces of all continuous *antilinear* functionals on $\Delta_{\pm}$), the triplets

$$\Delta_{\pm} \subset L^2(\mathrm{R}^+) \subset {}^{\times}(\Delta_{\pm}) \tag{2.3}$$

are RHS (Bohm and Gadella, 1989). Now, define $\boldsymbol{\Phi}^{\pm} = \boldsymbol{\Omega}^{\pm}U^{-1}\Delta_{\mp}$ [the change of sign is due to the usual notation in scattering theory; see Bohm and Gadella (1989)]. Then, the triplets

$$\boldsymbol{\Phi}^{\pm} \subset \mathcal{H}_{ac}(H) \subset {}^{\times}(\boldsymbol{\Phi}^{\pm}) \tag{2.4}$$

are new RHS. The relation between $\boldsymbol{\Phi}^{\pm}$ and $\mathcal{H}^2_{\mp} \cap S$ is given by

$$\boldsymbol{\Phi}^{\pm} \xrightarrow{(\boldsymbol{\Omega}^{\pm})^{-1}U} \Delta_{\mp} \xrightarrow{\theta_{\mp}^{-1}} \mathcal{H}^2_{\mp} \cap S \tag{2.5}$$

i.e., by the operator

$$W^{\pm} := (\boldsymbol{\Omega}^{\pm})^{-1}U\theta_{\mp}^{-1} \tag{2.6}$$

Then, if ψ^+ is an arbitrary vector in $\boldsymbol{\Phi}^+$ and φ^- is an arbitrary vector in $\boldsymbol{\Phi}^-$, we have, respectively,

$$W\psi^+ = \psi(E) \in S \cap \mathcal{H}^2; \qquad W^-\varphi^- = \varphi(E) \in S \cap \mathcal{H}^2_+ \tag{2.7}$$

We recall that the complex conjugate of a function in $\mathcal{H}^2_\pm$ is in $\mathcal{H}^2_\mp$, so that $\psi^\#(E) := [\psi(E)]^* \in S \cap \mathcal{H}^2_+$ and $\varphi^\#(E) := [\varphi(E)]^* \in S \cap \mathcal{H}^2_-$. Now, for any complex number z with imaginary part $\text{Im } z \geq 0$, the functional given by

$$\psi^+ \mapsto \langle\psi^+|z^+\rangle := \psi^\#(z) \tag{2.8}$$

is a continuous antilinear functional on $\mathbf{\Phi}^+$, where $\psi^\#(z)$ is the value of the function $\psi^\#(E)$ at the point z. This functional is an eigenvector of H with eigenvalue z [we use the same letter to denote the operator H and its extensions to the spaces ${}^\times(\mathbf{\Phi}^\pm)$], i.e., $H|z^+\rangle = z|z^+\rangle$. Also, for any z with $\text{Im } z \leq 0$, the functional given by

$$\varphi^- \mapsto \langle\varphi^-|z^-\rangle := \varphi^\#(z) \tag{2.9}$$

is a continuous antilinear functional on $\mathbf{\Phi}^-$, where $\varphi^\#(z)$ is the value of $\varphi^\#(E)$ at the point z. Also, $H|z^-\rangle = z|z^-\rangle$.

If simple poles of the S-matrix are located at $z_R = E_R - i\Gamma/2$ and $z_R^* = E_R + i\Gamma/2$ (second sheet), the vectors $|z_R^-\rangle \equiv |f_0\rangle$ and $|z_R^{*+}\rangle \equiv |\tilde{f}_0\rangle$ are called the decaying and growing Gamow vectors, respectively (Bohm and Gadella, 1989). They play the role of vector states for the decaying and growing parts of a resonance in a resonant scattering (Bohm, 1994; Bohm and Gadella, 1989).

Now, assume that a state created in the remote past as ψ^{in} produces a resonance and is transformed into $S\psi^{\text{in}} = \psi^{\text{out}}$ long after the interaction that produces the resonance is switched off. Since observations on ψ^{out} can be made on the region occupied by the registration apparatus only, what we measure is indeed the state φ^{out}, which is given by the projection of ψ^{out} into the region occupied by the registration apparatus. We further make an important Ansatz. Given ψ^{in}, the vector $\psi^+ := \mathbf{\Omega}^+\psi^{\text{in}}$ must be in $\mathbf{\Phi}^+$ and the vector ψ^{out} must have the property that $\varphi^- = \mathbf{\Omega}^-\psi^{\text{out}}$ in $\mathbf{\Phi}^-$. The consequence of this choice is that the sets of preparable and observable pure states belong to respective dense subspaces in $\mathcal{H}_{\text{ac}}$. This is not a limitation, since, given an arbitrary vector in $\mathcal{H}_{\text{ac}}$, we always can single out an arbitrarily close vector in a dense subspace. Thus, the error produced by this replacement can always be chosen smaller than the accuracy of the measurement apparatus. Now, following the standard literature (Bohm, 1981, 1994; Newton, 1982), we are interested in the transition amplitude between φ^{out} and $S\psi^{\text{in}}$:

$$\begin{aligned}(\varphi^{\text{out}}, S\psi^{\text{in}}) &= \int_0^\infty \langle\varphi^{\text{out}}|E\rangle S(E + i0)\langle E|\psi^{\text{in}}\rangle\, dE \\ &= \int_0^\infty \langle\varphi^-|E^-\rangle S(E + i0)\langle E^+|\psi^+\rangle\, dE \\ &= \int_{-\infty}^0 \langle\varphi^-|E^-\rangle S_{\text{II}}(E)\langle E^+|\psi^+\rangle\, dE + 2\pi i\{\text{Residue at } z_R\}\end{aligned} \tag{2.10}$$

Here, $S(E + i0)$ denotes the values of the S-matrix on the upper rim of the cut of the Riemann surface corresponding to the square root (Bohm, 1994) and $S_{\text{II}}(z)$ represent the values of the S-matrix on the second sheet. In particular $S_{\text{II}}(E)$ in (2.10) denotes the values of the S-matrix in the negative part of the real axis in the second sheet.

Now, let us make the important assumption that we have a pole of Nth order at z_R and another one at z_R^*. Then, the residue at z_R can be obtained as follows: The function $F(z) = \langle\varphi^-|z^-\rangle\langle z^{*+}|\psi^+\rangle$ is analytic on the lower half-plane and, hence, on a neighborhood of z_R, in which $S_{\text{II}}(z)$ admits the following Laurent series:

$$S_{\text{II}}(z) = \frac{a_1}{z - z_R} + \frac{a_2}{(z - z_R)^2} + \cdots + \frac{a_N}{(z - z_R)^N} + G(z) \qquad (2.11)$$

where $G(z)$ is analytic. The series expansion of $F(z)$ in this neighborhood is given by

$$F(z) = F(z_R) + F'(z_R)(z - z_R) + \frac{F''(z_R)}{2}(z - z_R)^2 + \cdots \qquad (2.12)$$

If we multiply (2.11) by (2.12), we obtain

$$\begin{aligned} F(z)S_{\text{II}}(z) & \\ &= \{R(z) + G(z)\}\left\{F(z_R) + \cdots + \frac{F^{(N-1}(z_R)}{(N-1)!}(z - z_R)^{N-1} + \cdots\right\} \\ &= F(z_R)R(z) + F'(z_R)(z - z_R)R(z) + \cdots \\ &\quad + \frac{F^{(N-1}(z_R)}{(N-1)!}(z - z_R)^{N-1}R(z) + \cdots + H(z) \end{aligned} \qquad (2.13)$$

where $H(z)$ is analytic on the considered neighborhood of z_R and

$$R(z) = \frac{a_1}{z - z_R} + \frac{a_2}{(z - z_R)^2} + \cdots + \frac{a_N}{(z - z_R)^N} \qquad (2.14)$$

From (2.13) and (2.14), we can obtain the residue of $S_{\text{II}}(z)F(z)$ at z_R, which is given by

$$F(z_R)a_1 + F'(z_R)a_2 + F''(z_R)\frac{a_3}{2} + \cdots + F^{(N-1}(z_R)\frac{a_N}{(N-1)!} \qquad (2.15)$$

Now, let us introduce the following notation:

$$\varphi^{\#}(z) = \langle\varphi^-|z^-\rangle; \qquad \psi(z) = \langle z^{*+}|\psi^+\rangle; \qquad F(z) = \varphi^{\#}(z)\psi(z) \qquad (2.16)$$

Using this notation, let us obtain the value of the kth derivative of $F(z)$ at the point z_R:

$$
\begin{aligned}
F^{(k}(z_R) &= \frac{d^k}{dz^k}\{\varphi^{\#}(z)\psi(z)\}_{z=z_R} \\
&= \left[\frac{d^k}{dz^k}\varphi^{\#}(z_R)\right]\psi(z_R) + k\left[\frac{d^{k-1}}{dz^{k-1}}\varphi^{\#}(z_R)\right]\left[\frac{d}{dz}\psi(z_R)\right] \\
&\quad + \cdots + \binom{k}{l}\left[\frac{d^{k-1}}{dz^{k-1}}\varphi^{\#}(z_R)\right]\left[\frac{d^l}{dz^l}\psi(z_R)\right] \\
&\quad + \cdots + \varphi^{\#}(z_R)\left[\frac{d^k}{dz^k}\psi(z_R)\right] \\
&= \sum_{l=0}^{k}\binom{k}{l}\left[\frac{d^{k-l}}{dz^{k-l}}\varphi^{\#}(z_R)\right]\left[\frac{d^l}{dz^l}\psi(z_R)\right]
\end{aligned} \tag{2.17}
$$

Since $\psi(E)$ and $\varphi^{\#}(E) = [\varphi(E)]^*$ belong to $\mathcal{H}^2_- \cap S$, their derivatives also belong to $\mathcal{H}^2_- \cap S$. For these derivatives, one obtains a formula analogous to the Cauchy formulas (Antoniou *et al.*, n.d.-): If Im $z < 0$, $\psi \in \mathcal{H}^2_- \cap S$, then

$$\psi^{(k}(z) = -\frac{k!}{2\pi i}\int_{-\infty}^{\infty}\frac{\psi(E)\,dE}{(E-z)^{k+1}} \tag{2.18}$$

An analogous formula is valid for $\varphi^{(n}(z) \in \mathcal{H}^2_+ \cap S$.

At this point, we need to define a new class of functionals which generalize (2.8) and (2.9). However, in our next definition, we shall restrict ourselves to a kind of physically relevant functional in correspondence with the poles z_R and z_R^* and the order of these poles. These functionals are the Gamow vectors for a resonant pole of order N. Thus, for any $k = 0, 1, 2, \ldots, N - 1$, and arbitrary $\psi^+ \in \mathbf{\Phi}^+$ and $\varphi^- \in \mathbf{\Phi}^-$, we set

$$\psi^+ \mapsto \langle\psi^+|\tilde{f}_k\rangle := \left[\frac{d^k}{dz^k}\psi^{\#}(z)\right]_{z=z_R^*}; \qquad \varphi^- \mapsto \langle\varphi^-|f_k\rangle := \left[\frac{d^k}{dz^k}\varphi^{\#}(z)\right]_{z=z_R} \tag{2.19}$$

Note that for $k = 0$ and $z = z_R^*$, we recover (2.8) from the first mapping in (2.19). Analogously, we recover (2.9) from the second mapping in (2.19) with $k = 0$ and $z = z_R$. To show that $|f_k\rangle \in {}^\times\mathbf{\Phi}^-$ and $|\tilde{f}_k\rangle \in {}^\times\mathbf{\Phi}^+$, one uses the fact that the kth derivative of a function in $\mathcal{H}^2_\pm \cap S$ is also in $\mathcal{H}^2_\pm \cap S$

(Antoniou *et al.*, n.d.-a). Also, using (2.18), it is shown immediately that

$$\left[\frac{d^k}{dz^k}\,\psi^{\#}(z)\right]_{z=z_R^*} = \left[\frac{d^k}{dz^k}\,\psi(z)\right]^*_{z=z_R}$$

$$\left[\frac{d^k}{dz^k}\,\varphi^{\#}(z)\right]_{z=z_R} = \left[\frac{d^k}{dz^k}\,\varphi(z)\right]^*_{z=z_R^*}; \qquad k = 0,1,2,\ldots, N-1 \tag{2.20}$$

so that, if we use the Dirac notation

$$\langle \tilde{f}_k|\psi^+\rangle := \langle \psi^+|\tilde{f}_k\rangle^*; \qquad \langle f_k|\varphi^-\rangle := \langle \varphi^-|f_k\rangle^* \tag{2.21}$$

we have

$$\left[\frac{d^k}{dz^k}\,\psi(z)\right]^*_{z=z_R} = \langle \tilde{f}_k|\psi^+\rangle; \qquad \left[\frac{d^k}{dz^k}\,\varphi(z)\right]^*_{z=z_R^*} = \langle f_k|\varphi^-\rangle,$$

$$k = 0, 1, 2, \ldots, N-1 \tag{2.22}$$

Using (2.19) and (2.22) in (2.17), we conclude that

$$F^{(k}(z_R) = \sum_{l=0}^{k} \binom{k}{l} \langle \varphi^-|f_{k-l}\rangle\langle \tilde{f}_l|\varphi^+\rangle \tag{2.23}$$

Now, let us come back to (2.10). Since $S = (\Omega^-)\dagger\Omega^+$, we have

$$(\varphi^{\text{out}}, S\psi^{\text{in}}) = (\varphi^-, \psi^+) \tag{2.24}$$

The vector ψ^+ is the decaying Gamow vector (Bohm, 1994). It represents the vector state resulting from the process in which a quasistationary state or resonance is formed up to time $t = 0$ and then starts to decay. If we carry (2.23) into (2.15), then (2.15) into (2.10), and then use (2.24), omitting the arbitrary vector φ^-, we get

$$\psi^+ = \int_{-\infty}^{0} |E^-\rangle S_{\text{II}}(E)\langle E^+|\psi^+\rangle\, dE + \sum_{l=0}^{k} c_k|f_k\rangle \tag{2.25}$$

where c_k are complex numbers depending on ψ^+. As we see, in the case of the presence of a pair of resonant poles of order N, the decaying state is the sum of the background, which is identical for any order of the poles, plus a linear combination of the vectors $|f_0\rangle, |f_1\rangle, \ldots, |f_{N-1}\rangle$, which are the decaying Gamow vectors for this situation. We are interested in some of the properties of these vectors and, in particular, the action of the Hamiltonian on them and their time evolution.

In order to obtain the action of H on the $|f_k\rangle$ we use the definition (1.2) for the extension of H to ${}^{\times}\mathbf{\Phi}^-$ (Bohm and Gadella, 1989). Let us consider the following bracket:

$$\langle H\varphi|f_k\rangle = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{D^k\{E\langle\varphi^-|E^-\rangle\}}{E-z_R}\,dE; \qquad D^k = \frac{d^k}{dE^k} \tag{2.26}$$

The identity in (2.26) is granted by the Tichmarsh theorem (Bohm and Gadella, 1989). Now, we perform the derivative under the integral sign

$$D^k\{E\langle\varphi^-|E\rangle\} = ED^k\{E\langle\varphi^-|E^-\rangle\} + kD^{k-1}\{E\langle\varphi^-|E^-\rangle\} \tag{2.27}$$

Thus, (2.26) and (2.27) give

$$\begin{aligned}\langle H\varphi^-|f_k\rangle &= \\ &= -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{ED^k\{E\langle\varphi^-|E^-\rangle\}}{E-z_R} \\ &\quad - \frac{k}{2\pi i}\int_{-\infty}^{\infty}\frac{D^{k-1}\{E\langle\varphi^-|E^-\rangle\}}{E-z_R} \\ &= z_R\langle\varphi^-|f_k\rangle + k\langle\varphi^-|f_{k-1}\rangle\end{aligned} \tag{2.28}$$

Then, the definition (2.21) implies that

$$H|f_k\rangle = z_R|f_k\rangle + k|f_{k-1}\rangle \tag{2.29}$$

This formula is true for $k = 1, 2, \ldots, N-1$. It is also correct for $k = 0$, although $|f_{k-1}\rangle$ does not exist in this case. Thus, in the subspace of ${}^{\times}\mathbf{\Phi}^-$ spanned by the vectors $|f_0\rangle$, $|f_1\rangle$, $\frac{1}{2}|f_2\rangle, \ldots, [1/(N-1)]|f_{N-1}\rangle$, the extended Hamiltonian has the following form:

$$H = \begin{pmatrix} z_R & 1 & 0 & \cdots & \cdots & 0 \\ 0 & z_R & 1 & \cdots & \cdots & 0 \\ 0 & 0 & z_R & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 0 & z_R & 1 \\ 0 & 0 & 0 & \cdots & 0 & z_R \end{pmatrix} \tag{2.30}$$

which exhibits the Jordan block form. Analogously, we obtain

$$H|\tilde{f}_k\rangle = z_R^*|\tilde{f}_k\rangle + k|\tilde{f}_{k-1}\rangle \tag{2.31}$$

Thus, in the subspace of ${}^{\times}\mathbf{\Phi}^{+}$ spanned by the vectors $|\tilde{f}_0\rangle$, $|\tilde{f}_1\rangle$, $\frac{1}{2}|\tilde{f}_2\rangle$, ..., $[1/(N-1)]|\tilde{f}_{N-1}\rangle$, the extended Hamiltonian has the following form:

$$H = \begin{pmatrix} z_R^* & 1 & 0 & \cdots & \cdots & 0 \\ 0 & z_R^* & 1 & \cdots & \cdots & 0 \\ 0 & 0 & z_R^* & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 0 & z_R^* & 1 \\ 0 & 0 & 0 & \cdots & 0 & z_R^* \end{pmatrix} \tag{2.32}$$

In order to obtain the time evolution of these Gamow vectors, we have to take into account that the adjoint of e^{-itH} given by e^{itH} fulfills the properties to be extended to the duals for some values of the time t only. In particular, e^{itH} reduces $\mathbf{\Phi}^{+}$, i.e., $e^{itH}\,\mathbf{\Phi}^{+} \subset \mathbf{\Phi}^{+}$ and is continuous on $\mathbf{\Phi}^{+}$ for $t < 0$ only. These properties are fulfilled as well on $\mathbf{\Phi}^{-}$ for $t > 0$ only (Bohm and Gadella, 1989). In this latter case, for an arbitrary $\varphi^{-} \in \mathbf{\Phi}^{-}$, we have

$$\langle e^{itH}\varphi^{-}|f_k\rangle = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{D^k\{e^{-itE}\langle\varphi^{-}|E^{-}\rangle\}}{E - z_R}\, dE \tag{2.33}$$

for $t > 0$, due to the Tichmarsh theorem (Bohm and Gadella, 1989). Since

$$D^k\{e^{-itE}\langle\varphi^{-}|E^{-}\rangle\} = \sum_{l=0}^{k}\binom{k}{l}(-it)^{k-l}e^{-itE}D^l\langle\varphi^{-}|E^{-}\rangle \tag{2.34}$$

Equations (2.33) and (2.34) finally give

$$e^{-itH}|f_k\rangle = \sum_{l=0}^{k}\binom{k}{l}(-it)^{k-l}e^{-itz_R}|f_l\rangle \qquad (t > 0) \tag{2.35}$$

Equation (2.35) can be written in matrix form. In this case, we can readily see that the matrix corresponding to the restriction of e^{-itH} to the subspace spanned by the vectors $|f_0\rangle$, $|f_1\rangle$, $\frac{1}{2}|f_2\rangle$, ..., $[1/(N-1)]|f_{N-1}\rangle$ is just the exponentiation of (2.30) given by

$$e^{-itH} = e^{-itz_R}\begin{bmatrix} 1 & -it & \dfrac{(-it)^2}{2} & \cdots & \dfrac{(-it)^{m-1}}{(m-1)!} \\ 0 & 1 & -it & \cdots & \dfrac{(-it)^{m-2}}{(m-2)!} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \qquad t > 0 \tag{2.36}$$

Similarly, for $t < 0$, we obtain

$$e^{-itH}|\tilde{f}_k\rangle = \sum_{l=0}^{k} \binom{k}{l} (-it)^{k-l} e^{itz_B^*} |\tilde{f}_l\rangle \qquad (t < 0) \tag{2.37}$$

so that, on the subspace spanned by the vectors $|\tilde{f}_0\rangle$, $|\tilde{f}_1\rangle$, $\frac{1}{2}|\tilde{f}_2\rangle$, ..., $[1/(N-1)]|\tilde{f}_{N-1}\rangle$, the time evolution has the following matrix form:

$$e^{-itH} = e^{-itz_R^*} \begin{bmatrix} 1 & -it & \frac{(-it)^2}{2} & \cdots & \frac{(-it)^{m-1}}{(m-1)!} \\ 0 & 1 & -it & \cdots & \frac{(-it)^{m-2}}{(m-2)!} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \qquad t < 0 \tag{2.38}$$

which is just the formal exponential of (2.32).

3. GENERALIZED SPECTRAL DECOMPOSITIONS

In this section, we present generalized spectral decompositions for the total Hamiltonian H and for the evolution semigroups. The clue which gives rise to these spectral decompositions is formula (2.10). To begin with, we shall assume that we have a pair of simple resonance poles on the second sheet. The generalization to a finite number of such poles is straightforward. If the number of poles is infinite, the nonintegral term in (2.10) becomes a series. There are some situations for which this series can be shown to converge (Gadella, 1997). For these cases, the generalization of the forthcoming formalism is pretty obvious. If we combine (2.20) with (2.24), we get

$$(\varphi^-, \psi^+) = \int_{-\infty}^{0} \langle \varphi^- | E^- \rangle S_{II}(E) \langle E^+ | \varphi^+ \rangle \, dE + a \langle \varphi^- | z_R^- \rangle \langle z_R^{*+} | \psi^+ \rangle \tag{3.1}$$

where $\psi^+ \in \Phi^+$ and $\varphi^- \in \Phi^-$. The integral term in (3.1) and all formulas derived from it will be henceforth denoted *background integrals*. The origin of this name is that the decaying vector ψ^+ is the sum of the contribution of the Gamow vector plus an extra term called the background that prevents ψ^+ from decaying exponentially.

Now, if we omit the vectors φ^- and ψ^+ in (3.1), absorb the irrelevant constant a in the definition of the Gamow vectors, and define $|E\rangle := \sqrt{S_{II}(E)}|E^-\rangle$ and $|\tilde{E}\rangle = \sqrt{S_{II}(E)}|E^+\rangle$, we obtain

$$I = \int_{-\infty}^{0} |E\rangle\langle \tilde{E}| \, dE + |f_0\rangle\langle \tilde{f}_0| \tag{3.2}$$

where the identity map I in (3.2) can be understood as the canonical embedding of Φ^+ into ${}^\times(\Phi^-)$. It is written as a sum of two continuous linear transformations from Φ^+ into ${}^\times(\Phi^-)$, as is not difficult to show. In this sense, it belongs to the space $\mathscr{L}(\Phi^+, {}^\times(\Phi^-)$ of continuous linear mappings from Φ^+ into ${}^\times\Phi^-$ (Schaeffer, 1970).

We recall now that H has the following important property: $H\Phi^\pm \subset \Phi^\pm$ (Bohm and Gadella, 1989) (we say that H reduces the spaces $\Phi^\pm$). As a consequence, $H\psi^+ \in \Phi^+$, for any $\psi^+ \in \Phi^+$, and we can replace ψ^+ by $H\psi^+$ in (3.1) so as to obtain the following spectral decomposition for H:

$$H = \int_{-\infty}^{0} E|E\rangle\langle\tilde{E}|\, dE + z_R|f_0\rangle\langle\tilde{f}_0| \tag{3.3}$$

For the same reason (Bohm and Gadella, 1989), if $t > 0$, we obtain the following decomposition for the evolution semigroup in the future:

$$U_t = e^{-itH} = \int_{-\infty}^{0} e^{-iEt}|E\rangle\langle\tilde{E}|\, dE + e^{-iz_Rt}|f_0\rangle\langle\tilde{f}_0|, \qquad t > 0 \tag{3.4}$$

In order to construct a spectral decomposition for the evolution semigroup for the past, we need to consider the complex conjugate of (1.4). This gives

$$I = \int_{-\infty}^{0} |\tilde{E}\rangle\langle E|\, dE + |\tilde{f}_0\rangle\langle f_0| \tag{3.5}$$

$$H = \int_{-\infty}^{0} E|\tilde{E}\rangle\langle E|\, dE + z_R^*|\tilde{f}_0\rangle\langle f_0| \tag{3.6}$$

For $t < 0$, the following expression is valid:

$$W_t = e^{-itH} = \int_{-\infty}^{0} e^{-iEt}|\tilde{E}\rangle\langle E|\, dE + e^{-itz_R^*}|\tilde{f}_0\rangle\langle f_0| \tag{3.7}$$

The operators in (3.5)–(3.7) are written in terms of sums of two operators in $\mathscr{L}(\Phi^-, {}^\times(\Phi^+))$.

Observe that, although (3.5)–(3.7) are the respective *formal* adjoints of (3.2)–(3.4), this *formal adjointness* does not mean that one operator is the adjoint of the other. In the case of I and H, they represent two different extensions of the same self-adjoint operator into two spaces that are, in some sense, complex conjugations of each other.

For the case of multiple-pole resonances, we obtain the following spectral decompositions for $t > 0$:

$$H = \int_{-\infty}^{0} E|E\rangle\langle\tilde{E}|\, dE + z_R \sum_{k=0}^{N-1} |f_k\rangle\langle\tilde{f}_k| + \sum_{s=0}^{N-2} |f_s\rangle\langle\tilde{f}_{s+1}| \qquad (3.8)$$

$$U_t := e^{-itH} = \int_{-\infty}^{0} e^{-iEt}|E\rangle\langle\tilde{E}|\, dE + e^{-iz_Rt}\Bigg[\sum_{k=0}^{N-1} |f_k\rangle\langle\tilde{f}_k|$$

$$- it|f_0\rangle\langle\tilde{f}_1| + \frac{(-it^2)^2}{2} |f_1\rangle\langle\tilde{f}_2|$$

$$+ \cdots + \frac{(-it)^{N-1}}{(N-1)!} |f_{N-2}\rangle\langle\tilde{f}_{N-1}|\Bigg] \qquad (3.9)$$

For $t < 0$, we obtain similar formulas.

So far, we have decomposed the Hamiltonian and the evolution semigroups in terms of operators of the form $|E\rangle\langle\tilde{E}|$, $|f_k\rangle\langle\tilde{f}_k|$, etc. They belong to the space $\mathcal{L}(\Phi^-, {}^{\times}(\Phi^+))$. Now we propose the possibility of introducing brackets among them. In the sequel we shall denote by $(\Phi^{\pm})^{\times}$ the spaces of continuous *linear* functionals on $\Phi^{\pm}$.

The derivation of these brackets is based on the following idea: The action of $|\tilde{E}\rangle$ on $\langle\psi^+|$ is a well-defined complex number we call $\langle\psi^+|\tilde{E}\rangle$. Since $|\psi^+\rangle \in \Phi^+ \subset {}^{\times}(\Phi^-)$ and $|\varphi^-\rangle \in \Phi^- \subset {}^{\times}(\Phi^+)$, the bracket $\langle\psi^+|\varphi^-\rangle$ is a bilinear form on the tensor product $(\Phi^+)^{\times} \otimes {}^{\times}(\Phi^-)$ with domain $\Phi^+ \otimes \Phi^-$. The possibility of forming the bracket $\langle\psi^+|\tilde{E}\rangle$ reveals that such a form can be extended.

If we apply the identity given by (3.5) to the left to $\langle\psi^+|$, we obtain the same vector, now considered as in $(\Phi^-)^{\times}$, and can write

$$\langle\psi^+|\tilde{E}\rangle = \langle\psi^+|I|\tilde{E}\rangle = \int_{-\infty}^{0} \langle\psi^+|\tilde{E}'\rangle\langle E'|\tilde{E}\rangle\, dE' + \langle\psi^+|\tilde{f}_0\rangle\langle f_0|\tilde{E}\rangle \qquad (3.10)$$

which implies that

$$\langle E'|\tilde{E}\rangle = \delta(E' - E); \qquad \langle f_0|\tilde{E}\rangle = 0 \qquad (3.11)$$

where the delta is defined with respect the integration from minus infinity to zero. Analogously, we use (3.2) to write

$$\langle\varphi^-|E\rangle = \langle\varphi^-|I|E\rangle = \int_{-\infty}^{0} \langle\varphi^-|E'\rangle\langle\tilde{E}'|E\rangle\, dE' + \langle\varphi^-|f_0\rangle\langle\tilde{f}_0|E\rangle \qquad (3.12)$$

to obtain

$$\langle\tilde{E}'|E\rangle = \delta(E' - E); \qquad \langle\tilde{f}_0|E\rangle = 0 \qquad (3.13)$$

If we now use the complex conjugates of (3.10) and (3.12), we get

$$\langle\tilde{E}|f_0\rangle = 0; \qquad \langle E|\tilde{f}_0\rangle = 0 \qquad (3.14)$$

Now, let us recall that $\langle\psi^+|\tilde{f}_0\rangle$ is a well-defined complex number. Using the same arguments as above, we have

$$\langle\psi^+|\tilde{f}_0\rangle = \int_{-\infty}^{0} \langle\psi^+|\tilde{E}\rangle\langle E|\tilde{f}\rangle\, dE + \langle\psi^+|\tilde{f}_0\rangle\langle f_0|\tilde{f}_0\rangle \tag{3.15}$$

Since $\langle E|\tilde{f}_0\rangle = 0$, we conclude that

$$\langle f_0|\tilde{f}_0\rangle = 1 \tag{3.16}$$

Analogously,

$$\langle\varphi^-|f_0\rangle = \int_{-\infty}^{0} \langle\varphi^-|E\rangle\langle\tilde{E}|f_0\rangle\, dE + \langle\varphi^-|f_0\rangle\langle\tilde{f}_0|f_0\rangle \tag{3.17}$$

implies

$$\langle\tilde{f}_0|f_0\rangle = 1 \tag{3.18}$$

From the mathematical point of view, the brackets in (3.11), (3.13), (3.14), (3.16), and (3.18) are well defined as kernels in the ordinary sense.

The construction of these brackets for simple pole resonances gives us the method of its generalization to multiple-pole resonances. In general we have for arbitrary $n = 0, 1, 2, \ldots, N - 1$

$$\begin{aligned} \langle\tilde{f}_n|\omega\rangle = \langle\omega|\tilde{f}_n\rangle = \langle f_n|\tilde{\omega}\rangle = \langle\tilde{\omega}|f_n\rangle = 0 \\ \langle f_n|\tilde{f}_m\rangle = \langle\tilde{f}_m|f_n\rangle = \delta_{m,n} \end{aligned} \tag{3.19}$$

Change of Limits in the Background Integral The background integral in the above formulas goes from $-\infty$ to 0. This may sometimes be an inconvenience. Take, for instance, formula (3.3) or (3.6). If we recall the traditional spectral decomposition theory in Hilbert space, these formulas may suggest that the spectrum of the Hamiltonian includes the negative real semiaxis. We know that this is wrong and that the spectrum of H is the positive semiaxis $\mathbb{R}^+$. Thus, it seems reasonable to change, whenever possible, the background integral for an integral from 0 to ∞. The properties of the Hardy functions with respect the Mellin transform (van Winter, 1974) give us the means to make this change. Let us consider the background integral in (3.1) and use the variable ω instead of E to avoid possible confusion. There exists a relation which gives the values of a Hardy function on the negative real semiaxis, knowing its values on $\mathbb{R}^+$ (Bohm and Gadella, 1989; van Winter, 1974). Using this relation, we have that

$$\int_{-\infty}^{0} \langle \varphi^- | \omega^- \rangle S_{\text{II}}(\omega) \langle \omega^+ | \psi^+ \rangle \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{0} S_{\text{II}}(\omega) \, d\omega \int_{-\infty}^{0} \omega^{-is-1/2} \, ds$$

$$\times \int_{0}^{\infty} \langle \varphi^- | E^- \rangle \langle E^+ | \psi^+ \rangle E^{is-1/2} \, dE \tag{3.20}$$

Let us consider now the following integral:

$$\int_{-\infty}^{0} S_{\text{II}}(\omega) \omega^{-is-1/2} \, d\omega \tag{3.21}$$

If we perform the change of variables (z real)

$$\omega = -e^z; \qquad d\omega = -e^z \, dz$$

$$\omega = -\infty \Rightarrow e^z = \infty \Rightarrow z = \infty$$

$$\omega = 0 \Rightarrow e^z = 0 \Rightarrow z = -\infty$$

the integral in (3.21) becomes

$$\int_{-\infty}^{\infty} S_{\text{II}}(-e^z) e^{z/2} e^{-isz} \, dz \tag{3.22}$$

If we write $g(z) = S_{\text{II}}(-e^z)e^{z/2}$, the latter integral is the Fourier transform of $g(z)$. The Fourier transform is well defined if $g(z)$ is square integrable. Then,

$$\int_{-\infty}^{\infty} |g(z)|^2 \, dz = \int_{-\infty}^{\infty} |S_{\text{II}}(-e^z)|^2 e^z \, dz = \int_{-\infty}^{0} |S_{\text{II}}(\omega)|^2 \, d\omega \tag{3.23}$$

This means that the Mellin transform (3.21) is well defined if $S(\omega) \in L^2(-\infty, \infty)$. Moreover,

$$\int_{-\infty}^{0} S_{\text{II}}(\omega) \omega^{-is-1/2} \, d\omega = \int_{-\infty}^{\infty} g(z) e^{-isz} \, dz \tag{3.24}$$

or

$$(\mathcal{M}S_{\text{II}})(s) = \sqrt{2\pi} (\mathcal{F}g)(s) \tag{3.25}$$

where $(\mathcal{M}S_{\text{II}})(s)$ denotes the Mellin transform of $S_{\text{II}}(\omega)$ (van Winter, 1974) and $(\mathcal{F}g)(s)$ the Fourier transform of $g(z)$. Also, $S_{\text{II}}(\omega) \in L^2(-\infty, 0) \Leftrightarrow g(z)$

$\in L^2(-\infty, \infty) \Rightarrow (\mathcal{M}S_{II})(s) \in L^2(-\infty, \infty)$. Thus, we can write the background integral (3.20) as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} (\mathcal{M}S_{II})(s)E^{is-1/2}\, ds \int_0^{\infty} \langle\varphi^-|E^-\rangle\langle E^+|\psi^+\rangle\, dE$$
$$= \int_0^{\infty} \langle\varphi^-|E^-\rangle G(E)\langle E^+|\psi^+\rangle\, dE \qquad (3.26)$$

where

$$G(E) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (\mathcal{M}S_{II})(s)E^{is-1/2}\, ds \qquad (3.27)$$

is the inverse Mellin transform of $(\mathcal{M}S_{II})(-s)$ and is therefore an $L^2(0, \infty)$ function. Since $\langle\varphi^-|E^-\rangle$ and $\langle E^+|\varphi^+\rangle$ are the values of these functions on the positive semiaxis, we have that for these $|E^\pm\rangle$

$$H|E^\pm\rangle = E|E^\pm\rangle \qquad (3.28)$$

We have seen that the Mellin transform does the job of replacing the integration limits in the background integral. The new integration limits cause the integral to be extended over the positive semiaxis, which coincides with the spectrum of the total Hamiltonian. In the new notation, the integral in formulas (3.3) and (3.6) is taken from 0 to ∞ and the vectors $|E\rangle$ and $|\tilde{E}\rangle$ are replaced by $|E\rangle = \sqrt{G(E)}|E^-\rangle$ and $|\tilde{E}\rangle = \sqrt{G(E)}|E^+\rangle$ with $E > 0$.

4. GAMOW STATES AND THEIR TIME EVOLUTION

Our next goal is to define the evolution of the so-called Gamow states, which are the state operators corresponding to Gamow vectors. Since Gamow vectors are not in Hilbert space, their corresponding "densities" are not operators on Hilbert space. We have already mentioned that objects like $|f_k\rangle\langle\tilde{f}_k|$, etc., belong to spaces of operators like $\mathcal{L}(\Phi^+, {}^\times\Phi^-)$. Here, we wish to understand these objects from another point of view. The point of departure of this understanding is the "rigging" of the Liouville space.

The Liouville space is the space of states in statistical mechanics. For the case of quantum statistical mechanics, the space of states is usually chosen to be the tensor product $\mathcal{H} \otimes \mathcal{H}^\times$, where $\otimes$ denotes the tensor product of Hilbert spaces and $\mathcal{H}^\times$ the dual space of the infinite-dimensional separable Hilbert space $\mathcal{H}$. Henceforth, we shall look at $\mathcal{H}^\times$ as different from $\mathcal{H}$ in some sense. Vectors in $\mathcal{H}^\times$ are somewhat complex conjugate of vectors in $\mathcal{H}$. Let us explain this idea more in detail. Under the conditions given in Section 2, there are unitary mappings $V^\pm = \Omega^\pm U$ from $\mathcal{H}$ to $L^2(\mathbb{R}^+)$. The dual space of $L^2(\mathbb{R}^+)$ can be looked on as the space of all complex conjugates of functions in $L^2(\mathbb{R}^+)$ (which is of course the same space). This follows from the definition of the scalar product in $L^2\mathbb{R}^+$):

$$(f, g) = \int_0^\infty f^*(E)g(E)\, dE \tag{4.1}$$

We see here that the role of a function depends on the fact that it belongs to the Hilbert space or to its dual. This idea is immediately translated to $\mathcal{H}$ via the unitary operators $V^{\pm}$. This trivial consideration is however, useful, in order to choose the space of test vectors for the extended or "rigged" Liouville space. In fact, the space of complex conjugate vectors of $\mathbf{\Phi}^{\pm}$ is, according to the above remarks, $\mathbf{\Phi}^{\mp}$. Therefore, we can propose the following spaces of test vectors $\mathbf{\Phi}^{\pm} \otimes \mathbf{\Phi}^{\mp}$ in order to construct the following triplets:

$$\mathcal{L}^{\pm} := \mathbf{\Phi}^{\pm} \otimes \mathbf{\Phi}^{\mp} \subset \mathcal{H} \otimes \mathcal{H}^{\times} \subset (\mathbf{\Phi}^{\pm})^{\times} \otimes {}^{\times}(\mathbf{\Phi}^{\mp}) =: (\mathcal{L}^{\pm})^{\times} \tag{4.2}$$

where $(\mathbf{\Phi}^{\pm})^{\times}$ are the spaces of continuous *linear* functionals on $\mathbf{\Phi}^{\pm}$, respectively.

Remark Concerning Notation We denote the action of $|F^{\pm}\rangle\langle G^{\mp}| \in (\mathcal{L}^{\pm})^{\times}$ into $|\varphi^{\pm}\rangle\langle\psi^{\mp}| \in \mathcal{L}^{\pm}$ as

$$(|F^{\pm}\rangle\langle G^{\mp}|, |\varphi^{\pm}\rangle\langle\psi^{\mp}|) = \langle F^{\pm}|\varphi^{\pm}\rangle\langle\psi^{\mp}|G^{\mp}\rangle \tag{4.3}$$

Density Operators for the Gamow States The second goal of the present section consists in defining density operators for the Gamow states in a similar way in which we construct density operators for pure states in Hilbert space. To begin with, we first consider simple pole resonances. The density operators for multiple-pole resonances will be constructed later in this section. First, we note that since Gamow vectors do not belong to Hilbert space, their corresponding densities cannot be traciable operators. However, they can be defined as objects in the dual spaces $(\mathcal{L}^{\pm})^{\times}$, exactly as Gamow vectors have been defined in certain dual spaces. It is also reasonable that the Gamow densities have the same exponential behavior with respect to time as their corresponding Gamow vectors. Let us choose

$$\rho = |\tilde{f}_0\rangle\langle f_0| \in (\mathcal{L}^{+})^{\times} \qquad \text{for the decaying Gamow state}$$

$$\tilde{\rho} = |f_0\rangle\langle \tilde{f}_0| \in (\mathcal{L}^{-})^{\times} \qquad \text{for the Growing Gamow state} \tag{4.4}$$

Now, we need to find a reasonable time evolution for the densities ρ and $\tilde{\rho}$. By analogy with the Gamow vectors and the usual Liouville space, one may think that the time evolution of these states should be provided by certain extensions of the exponential of the Liouville operator $e^{-it\mathrm{L}} = U_t \otimes U_t^{\dagger} =$

$e^{-itH} \otimes e^{itH}$. However, this operator cannot be extended by continuity either to $(\mathcal{L}^+)^\times$ nor to $(\mathcal{L}^-)^\times$, since it does not reduce any of the spaces $\mathcal{L}^\pm$. Therefore, it is nonsense to apply it to ρ or to $\tilde{\rho}$ for any $t \neq 0$. For instance, in the decay process, in order to define $\rho(t)$ for $t > 0$, one may try $[\rho(0) = |\tilde{f}_0\rangle\langle f_0|]$

$$\begin{aligned}(\rho(t), |\psi^+\rangle\langle\varphi^-|) &= (e^{-it\mathrm{L}}\rho(0), |\psi^+\rangle\langle\varphi^-|)\\ &= (\rho(0), e^{it\mathrm{L}}|\psi^+\rangle\langle\varphi^-|)\\ &= (\rho(0), U_t^\dagger|\psi^+\rangle\langle\varphi^-|U_t)\\ &= \langle\tilde{f}_0|U_t^\dagger|\psi^+\rangle\langle\varphi^-|U_t|f_0\rangle\end{aligned} \tag{4.5}$$

If $U_t = e^{-itH}$, the bracket $\langle\tilde{f}_0|U_t^\dagger|\psi^+\rangle$ is not well defined because $U_t^\dagger = \mathrm{e}^{-itH}$ does not map $\mathbf{\Phi}^+$ into itself. However, if we choose another conjugation for U_t different from the standard adjointness operation, it would be possible to give a meaning to $\langle\tilde{f}_0|U_t^\dagger|\psi^+\rangle$. This can be done with the aid of the spectral decompositions given in Section 3. With these ideas in mind, take the formal conjugate in (3.4) as follows:

$$U_t^* = \int_{-\infty}^0 e^{iEt}|E\rangle\langle\tilde{E}|dE + e^{iz_R^*t}|f_0\rangle\langle\tilde{f}_0| \qquad (t > 0) \tag{4.6}$$

Indeed, (4.4) is a generalized sum of operators in $\mathcal{L}(\mathbf{\Phi}^+, {}^\times\mathbf{\Phi}^-)$ giving another operator in this space. If we perform complex conjugation on the coefficients of this sum, we obtain (4.6). Now, after (4.6), we get

$$\langle\tilde{f}_0|U_t^*|\psi^+\rangle = \int_{-\infty}^0 e^{iEt}\langle\tilde{f}_0|E\rangle\langle\tilde{E}|\psi^+\rangle\, dE + e^{iz_R^*t}\langle\tilde{f}_0|f_0\rangle\langle\tilde{f}_0|\psi^+\rangle = e^{iz_R^*t}\langle\tilde{f}_0|\psi^+\rangle \tag{4.7}$$

On the other hand, for $t > 0$, $\langle\varphi^-|U_t|f_0\rangle = \langle\varphi^-|e^{-itH}|f_0\rangle$ is well defined and is equal to $e^{-itz_R}\langle\tilde{f}_0|\varphi^+\rangle$. Now, we replace $\langle\tilde{f}_0|U_t^\dagger|\psi^+\rangle$ by $\langle\tilde{f}_0|U_t^*|\psi^+\rangle$ to obtain

$$(\rho(t), |\psi^+\rangle\langle\varphi^-|) = e^{-t\Gamma}\langle\tilde{f}_0|\psi^+\rangle\, \langle\psi^-|f_0\rangle = e^{-t\Gamma}(\rho(0), |\psi^+\rangle\langle\varphi^-|), \qquad t > 0 \tag{4.8}$$

Since $|\psi^+\rangle\langle\varphi^-|$ is arbitrary,

$$\rho(t) = e^{-t\Gamma}\rho(0); \qquad t > 0 \tag{4.9}$$

This trick may have, however, a major inconvenience: it is not difficult to see that U_t^* as in (4.6) is not an extension of $U_t^\dagger$ and therefore we are using an extended dynamics which is not a natural extension of the usual dynamics in Hilbert space. Our approach is mainly justified by the fact that it gives the correct exponential decay for the decaying Gamow state (as it gives the correct exponential growth for the growing Gamow state). This inconvenience shows that something deep is really behind the extension of the Gamow vector formalism to the rigged Liouville space, which will be investigated in elsewhere (Antoniou *et al.*, n.d.-b).

For $t < 0$ and $\tilde{\rho}(0) := \rho$ as in (4.5), we have an analogous situation. The equivalent of (4.6) is now

$$(\tilde{\rho}(t), |\varphi^-\rangle\langle\psi^+|) = \langle f_0|W_t^\dagger|\varphi^-\rangle\langle\psi^+|W_t|\tilde{f}_0\rangle \tag{4.10}$$

where $W_t = e^{-itH}$. For $t < 0$, the identity $\langle\psi^+|W_t|\tilde{f}_0\rangle = e^{-itz_R^*}\langle\psi^+|\tilde{f}_0\rangle$ does not represent a mathematical problem, but, again, $\langle\tilde{f}_0|W_t^\dagger|\varphi^-\rangle$ is not defined. To avoid this inconvenience, we proceed in analogy to what we have done for the decay process by introducing the conjugate W_t^* of W_t in (3.7) as

$$W_t^* = \int_{-\infty}^0 e^{iEt}|\tilde{E}\rangle\langle E|dE + e^{itz_R}|\tilde{f}_0\rangle\langle f_0| \tag{4.11}$$

and then replacing $\langle f_0|W_t^\dagger|\varphi^-\rangle$ by $\langle f_0|W_t^*|\varphi^-\rangle$. Thus, we have

$$\langle f_0|W_t^*|\varphi^-\rangle = e^{itz_R}\langle f_0|\varphi^-\rangle \tag{4.12}$$

so that for $t < 0$

$$(\tilde{\rho}(t), |\varphi^-\rangle\langle\psi^+|) = e^{\Gamma t}(\tilde{\rho}(0), |\varphi^-\rangle\langle\psi^+|) \tag{4.13}$$

which can be written as

$$\tilde{\rho}(t) = e^{\Gamma t}\tilde{\rho}(0), \qquad t < 0 \tag{4.14}$$

After all of the above, we realize that we can no longer write $\rho(t) = e^{-itL}\rho(0)$, $t > 0$, and $\tilde{\rho}(t) = e^{-itL}\tilde{\rho}(0)$, $t < 0$, and keeping at the same time the usual meaning of the exponential e^{-itL}. From now on, we shall write $\rho(t) = \mathcal{U}_t\rho(0)$, $t > 0$, and $\tilde{\rho}(t) = \mathcal{W}_t\tilde{\rho}(0)$, $t < 0$. We want to show that there exists a relation between the semigroups $\mathcal{U}_t$ and $\mathcal{W}_t$. In order to do it, let us define the notion of formal adjoint of U_t, U_t^*, W_t, and W_t^*. For U_t, its formal adjoint $U_t^\dagger$ is given by

$$\langle\varphi^-|U_t|\psi^+\rangle^* = \langle\psi^+|U_t^\dagger|\varphi^-\rangle \tag{4.15}$$

so that (4.15) along with (3.4) gives

$$U_t^\dagger = e^{itz_R^*}|\tilde{f}_0\rangle\langle f_0| + \int_{-\infty}^0 e^{iEt}|\tilde{E}\rangle\langle E|dE \qquad (t > 0) \tag{4.16}$$

Similarly, one has

$$U_t^{*\dagger} = e^{-iz_R t}|\tilde{f}_0\rangle\langle f_0| + \int_{-\infty}^{0} e^{-iEt}|\tilde{E}\rangle\langle E|dE \qquad (t > 0) \tag{4.17}$$

$$W_t^{\dagger} = e^{iz_R t}|\tilde{f}_0\rangle\langle f_0| + \int_{-\infty}^{0} e^{iEt}|E\rangle\langle\tilde{E}|dE \qquad (t > 0) \tag{4.18}$$

$$W_t^{*\dagger} = e^{-iz_R^* t}|f_0\rangle\langle\tilde{f}_0| + \int_{-\infty}^{0} e^{-iEt}|E\rangle\langle\tilde{E}|dE \qquad (t > 0) \tag{4.19}$$

Note that $U_t^{*\dagger} = U_t^{\dagger *}$ and $W_t^{*\dagger} = W_t^{*\dagger}$. Then, for $t > 0$,

$$(\rho(t), |\psi^+\rangle\langle\varphi^-|) = (\mathcal{U}_t\rho, |\psi^+\rangle\langle\varphi^-|) = \langle\tilde{f}_0|U_t^*|\psi^+\rangle\langle\varphi^-|U_t|f_0\rangle$$
$$= (U_t^{*\dagger}|\tilde{f}_0\rangle\langle f_0|U_t^{\dagger}, |\psi^+\rangle\langle\varphi^-|) \tag{4.20}$$

Thus,

$$\mathcal{U}_t\rho = U_t^{*\dagger}\rho U_t^{\dagger} \qquad \text{or} \qquad \mathcal{U}_t = U_t^{*\dagger} \otimes U_t^{\dagger} \tag{4.21}$$

Analogously, for $t < 0$,

$$\mathcal{W}_t\tilde{\rho} = W_t^{*\dagger}\tilde{\rho}W_t^{\dagger} \qquad \text{or} \qquad \mathcal{W}_t = W_t^{*\dagger} \otimes W_t^{\dagger} \tag{4.22}$$

If we compare (3.4) with (4.20) and (4.6) with (4.21), we obtain

$$W_t^{\dagger} = U_{-t}; \qquad W_t^{*\dagger} = U_{-t}^{*} \tag{4.23}$$

Now, let us define the formal adjoint $\mathcal{U}_{-t}^{\dagger}$ as

$$\mathcal{U}_{-t}^{\dagger} := (U_{-t}^{*\dagger} \otimes U_{-t}^{\dagger})^{\dagger} = U_{-t}^{*} \otimes U_{-t} = W_t^{*\dagger} \otimes W_t^{\dagger} \tag{4.24}$$

so that

$$\mathcal{W}_t = \mathcal{U}_{-t}^{\dagger} \tag{4.25}$$

The operators U_t, U_t^*, W_t, W_t^* as well as their adjoints are well defined for *all* values of t. However, in principle, they cannot be extended by continuity to the duals $^{\times}\mathbf{\Phi}^-$ and $^{\times}\mathbf{\Phi}^+$. Therefore, $\mathcal{U}_t$ and $\mathcal{W}_t$ are not continuous operators on $(\mathbf{\Phi}^-)^{\times} \otimes {}^{\times}\mathbf{\Phi}^+$ and $(\mathbf{\Phi}^+)^{\times} \otimes {}^{\times}\mathbf{\Phi}^-$, respectively. Continuity is not essential here, since these operators have a domain which is bigger than $\mathcal{H} \otimes \mathcal{H}^{\times}$ and includes the Gamow states. In a subsequent paper, we shall introduce a mathematical apparatus to make them continuous operators on a space of states that includes Gamow states (Antoniou *et al.*, n.d.-b).

Important Remark The fact that $\mathcal{U}_t$ and $\mathcal{W}_t$ are defined for *all* values of time, so that they act on the Gamow states for all times as

$$\mathcal{U}_t|\tilde{f}_0\rangle\langle f_0| = e^{-\Gamma t}|\tilde{f}_0\rangle\langle f_0| \qquad (t \in \mathbb{R})$$

$$\mathcal{W}_t|f_0\rangle\langle \tilde{f}_0| = e^{\Gamma t}|f_0\rangle\langle \tilde{f}_0| \qquad (t \in \mathbb{R}) \tag{4.26}$$

has important consequences. The traditional formalism of Gamow vectors says that $|f_0\rangle$ is created at $t = 0$ and evolves for positive times and $|\tilde{f}_0\rangle$ disappears at $t = 0$ and its evolution has taken place for negative values of time. This implies a *choice of the origin of time* and consequently a *violation of temporal inhomogeneity.* This violation does not take place with the present point of view, because no matter at which time we start our measurements on a radiative sample, we always observe exponential decay. We acknowledge I. Antoniou for this remark.

5. TIME EVOLUTION FOR MULTIPLE-POLE RESONANCES

In this section we generalize the previous study to multiple-pole resonances. We start with the discussion on the simplest situation of having double-pole resonances; the extension to the higher order case is straightforward. Here, decaying Gamow states are linear combinations of dyads of the form $|\tilde{f}_i\rangle\langle f_k|$ $(i, j = 0, 1)$, whose coefficients have some properties to be specified later. Analogously, growing Gamow states are linear combination of $|f_k\rangle\langle \tilde{f}_j|$. Let us study the decaying process first. The evolution semigroup e^{-itH} for $t > 0$ has, after (2.36), the following expression:

$$U_t = e^{-itH} = \begin{pmatrix} e^{-itz_R} & -ite^{-itz_R} \\ 0 & e^{-itz_R} \end{pmatrix} + \int_{-\infty}^{0} e^{-iEt}|E\rangle\langle \tilde{E}|dE \tag{5.1}$$

In analogy with what we have done before, we define the conjugate U_t^* of U_t as

$$U_t^* = \begin{pmatrix} e^{itz_R^*} & 0 \\ ite^{itz_R^*} & e^{itz_R^*} \end{pmatrix} + \int_{-\infty}^{0} e^{iEt}|E\rangle\langle \tilde{E}|dE \tag{5.2}$$

For our purposes, only the matrix part in (5.1) and (5.2) is relevant. Therefore, a convenient matrix notation will be useful here and will simplify our calculations. Thus, we shall use

$$|f_0\rangle\langle \tilde{f}_0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \qquad |f_0\rangle\langle \tilde{f}_1| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$|f_1\rangle\langle \tilde{f}_0| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \qquad |f_1\rangle\langle \tilde{f}_1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.3}$$

After this identification, the matrix parts for U_t and U_t^* can be written, respectively, as

$$e^{-itz_R}|f_0\rangle\langle\tilde{f}_0| - ite^{-itz_R}|f_0\rangle\langle\tilde{f}_1| + e^{-itz_R}|f_1\rangle\langle\tilde{f}_1| \tag{5.4}$$

$$e^{itz_R^*}|f_0\rangle\langle\tilde{f}_0| + ite^{itz_R^*}|f_1\rangle\langle\tilde{f}_0| + e^{itz_R^*}|f_1\rangle\langle\tilde{f}_1| \tag{5.5}$$

As we have mentioned, decaying Gamow states should be linear combinations of dyads of Gamow vectors:

$$\rho = \sum_{i,j=0}^{1} \rho_{ij}|\tilde{f}_i\rangle\langle f_j| \tag{5.6}$$

where ρ_{ij}, i, $j = 0, 1$, are complex numbers. We study the time evolution of ρ as we have done for the case of a simple pole resonance. For any dyad $|\psi^+\rangle\langle\varphi^-|$, we write

$$\begin{aligned}(\rho(t), |\psi^+\rangle\langle\varphi^-|) &= (\mathcal{U}_t\rho, |\psi^+\rangle\langle\varphi^-|) = (\rho, U_t^*|\psi^+\rangle\rangle\varphi^-|U_t) \\ &= \sum_{i,j=0}^{1} \rho_{ij}\langle\tilde{f}_j|U_t^*|\psi^+\rangle\langle\varphi^-|U_t|f_i\rangle \end{aligned} \tag{5.7}$$

To obtain the sum in (5.7), one should replace U_t and U_t^* by (5.4) and (5.5), respectively. For practical purposes, it is easier to evaluate directly $U_t|f_k\rangle\langle\tilde{f}_l|U_t^*$ using the matrix notation suggested above. We only write here the final result, in which $t > 0$, as

$$U_t|f_1\rangle\langle\tilde{f}_1|U_t^* = e^{-\Gamma t}|f_1\rangle\tilde{f}_1| \tag{5.8}$$

$$U_t|f_0\rangle\langle\tilde{f}_1|U_t^* = e^{-\Gamma t}\{|f_0\rangle\langle\tilde{f}_1| + it|f_1\rangle\langle\tilde{f}_1|\} \tag{5.9}$$

$$U_t|f_1\rangle\langle\tilde{f}_0|U_t^* = e^{-\Gamma t}\{|f_1\rangle\langle\tilde{f}_0| - it|f_1\rangle\langle\tilde{f}_1|\} \tag{5.10}$$

$$U_t|f_0\rangle\langle\tilde{f}_0|U_t^* = e^{-\Gamma t}\{|f_0\rangle\langle\tilde{f}_0| - it|f_0\rangle\langle\tilde{f}_1| + it|f_1\rangle\langle\tilde{f}_0| - t^2|f_1\rangle\langle\tilde{f}_1|\} \tag{5.11}$$

Formulas (5.8)–(5.11) show that (5.7) can be written as

$$(\rho(t), |\psi^+\rangle\langle\varphi^-|) = \sum_{i,j=0}^{1} \rho_{ij}(t)\langle\tilde{f}_i|\psi^+\rangle\langle\varphi^-|f_j\rangle \tag{5.12}$$

where

$$\begin{aligned} \rho_{00}(t) &= e^{-\Gamma t}\rho_{00}, \qquad \rho_{01}(t) = e^{-\Gamma t}(\rho_{01} - it\rho_{00}) \\ \rho_{10}(t) &= e^{-\Gamma t}(\rho_{10} + it\rho_{00}), \qquad \rho_{11}(t) = e^{-\Gamma t}(\rho_{11} - it\rho_{10} + it\rho_{01} + t^2\rho_{11}) \end{aligned} \tag{5.13}$$

Note that $\rho_{ij} = \rho_{ij}(0)$. For $t < 0$, one finds analogously,

$$\tilde{\rho}(t) = \mathcal{W}_t\tilde{\rho}(0) = \sum_{i,j=0}^{1} \tilde{\rho}_{ij}(t)|f_i\rangle\langle\tilde{f}_j| \tag{5.14}$$

where $\tilde{\rho}_{ij}(t)$ have the same explicit form as ρ_{ij}, only replacing $e^{-\Gamma t}$ with $t > 0$ by $e^{\Gamma t}$ with $t < 0$.

For an Nth-pole resonance the generalization of the above formalism is straightforward. The results that can be obtained here are an immediate generalization of those found for $N = 2$. In particular, the time evolution for $|\tilde{f}_i\rangle\langle f_k|$ or $|f_k\rangle\langle\tilde{f}_j|$ is not exponential except for $|f_{N-1}\rangle\langle\tilde{f}_{N-1}|$ and $|\tilde{f}_{N-1}\rangle\langle f_{N-1}|$. As an example, we readily get ($t > 0$)

$$\mathcal{U}_t|\tilde{f}_0\rangle\langle f_0| = e^{-\Gamma t} \sum_{k,l=0}^{N-1} \rho_{kl}|\tilde{f}_k\rangle\langle f_l| \tag{5.15}$$

where

$$\rho_{kl} = \frac{(it)^{k-1}(-it)^{l-1}}{(k-1)!(l-1)!} \tag{5.16}$$

Again, Gamow states are suitable linear combinations of dyads of Gamow vectors. The coefficients of these linear combinations, which are matrix elements in matrix notation, are functions of $t > 0$ for the decaying process and of $t < 0$ for the growing process. For the decaying process these functions are $\rho_{kl}(t) = e^{-\Gamma t}P_{k+l}(t)$, where P_{k+l} is a polynomial of order $k + l$. A similar result is obtained for the growing process, just replacing $e^{-\Gamma t}$ by $e^{\Gamma t}$.

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